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Estimation of Parameters of the Displaced Exponential Distribution Using the Maximum Likelihood-Cumulative Distribution Function Technique

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ABSTRACT

The displaced exponential distribution was found to be a good model for the distribution of traffic time gaps and headways in New Zealand by Ashton (1975). Cabanlit (1993), however, found that by using the same moment estimators for the parameters of the displaced exponential law, the estimators behaved quite badly both from the bias and MSE points of view. It is shown in this paper that by using a mixed MLE-cdf technique of estimation, the parameters of the displaced exponential distribution may be estimated with greater accuracy Theoretical results are established which are supported by a small Monte Carlo experiment.

Keywords: Displaced exponential distribution; maximum likelihood estimator; moment estimator; cdf-estimator; asymptotically normal; bias; mean-squared error.

1. Introduction

The distribution of time gaps between successive vehicles in a traffic is of interest in applied work. To this end, Ashton (1975) suggested three possible distributions for time gaps. These distributions are the displaced exponential distribution, Schull's composite distribution and the semi-Poisson distribution. Of these three distributions, the displaced exponential given by

$$f(t) = (1/\beta)\, e^{-(t-\alpha)/\beta}, \quad \beta \geq 0, t \geq \alpha \tag{1}$$

is of interest in this paper.

Ashton and Burkley (1976) claimed that the displaced exponential distribution explains data behavior when traffic is from moderate to heavy (less than 800 vehicles per hour). Cabanlit (1993), however, found that by using the same moment estimators for the parameters of the displaced exponential, data obtained from selected traffic intersection in Cagayan de Oro do not lend support to the Ashton hypothesis. The Schull's composite distribution is a good model for describing traffic flows in metropolitan areas where the composition of vehicles passing through a given street is rather varied.

It is well known in estimation theory that the method of moment estimators are not generally, good estimators in the sense of the classical criteria of unbiasedness, asymptotic efficiency nor minimum variance. It is therefore conjectured that the lack of fit observed by Cabanlit(1993) is not really due to wrong model specification but rather because of the poor choice of an estimation procedure. Cabanlit's method of moments estimators proceeds by equating the sample moments obtained from the data to the theoretical moments of the displaced exponential distribution.

This paper concentrates on the estimation problem for a displaced exponential distribution using the method of maximum likelihood and a technique developed by the author using the observed cumulative distribution function for traffic time gaps. The

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properties of the MLE have not been explored in Ashton(1975) hence its inclusion in the present study.

2. The Maximum Likelihood Estimators

The likelihood function associated with equation (1) is given by:

$$L = 1/\beta^{n} \exp\left(-\sum (t_i - \alpha)\right)/\beta \qquad (2)$$

Maximum L with respect to α and β yields the following likelihood estimators:

$$\hat{\alpha} = \min\{ti\} \hat{\beta} = \sum (ti - \alpha)/n \text{ ....} \qquad (3)$$

The following theorems specify the properties of these estimators:

Theorem 1: *The estimator* $\hat{\alpha}$ = *min* $\{t_i\}$*of the displaced exponential distribution is asymptotically unbiased, that is:* $E(\hat{\alpha})$ *tends to* α *as n tends to infinity.*

Proof:

The density function of the minimum of the observed time gaps is:

$$g(y) = (n/\beta)\exp(-n(y-\alpha)/\beta$$

Thus,

$$E(y) = \int_{\alpha}^{\infty} (y/\beta)\exp(-n(y-\alpha))dy$$

$$= \alpha + (\beta/n)$$

Let $t\varphi$ be a point such that F_n $(t\varphi)=\varphi$, where $0<\varphi<1$. Then $t\varphi$ is called the upper φth quantile of $F_n(t)$.

The cdf-technique starts by estimating t from the sample distribution function. The estimator $t_{(\varphi)}$ of $t\varphi$ will be an order statistics from the underlying distribution F(t). For a one-parameter family { F(t, θ)}, the choice of φ is usually dictated by some practical considerations. The parameter φ is then estimated by solving the equation:

$$F(t_{(\varphi)},\theta) = \varphi \text{, or equivalently;} \qquad (4)$$

$$\int_{\varphi}^{t(\ )} f(t,\theta)dt = \varphi \quad \text{if } F(t,0) \text{ has a density.} \qquad (5)$$

From equation (4) it follows that:

$$\theta = F^{-1}(t_{(\varphi)}, \varphi). \qquad (6)$$

Equation (6) states that the estimator of θ is a function of the order statistics $t_{(\varphi)}$ and φ.

Ashley (1975), for instance, claims that when the traffic consists of two types of vehicles: normal and heavy, then the time-gaps less than or equal to 4.5 seconds are considered heavy flow while those over 4.5 seconds are considered normal flow. These observations allow one to estimate φ and $t_{(\varphi)}$ from the data.

Example: Suppose that for a given dataset a δ = 80 percent of the time-gaps is less than or equal to 4.5 seconds. Here, the random variable is δ (t_δ = 4.5 seconds by definition). Assume that a displaced exponential distribution is to be fitted.

Theorem 2: *If α=0 then $\hat{\beta}$ = t is unbiased for B. Moreover, β is UMVUE.*

Proof:

Let $\phi(s)$ be the moment -generating function of t, it follows that;

$$\phi_{(s)}=E\{\exp(st)\}=1/(s\beta-1)$$

the moment generating function of $n\hat{\beta}$ is

$$\phi_{(s)} = 1/(s\beta-1)$$

If follows that $n\hat{\beta}$ is Gamma distributed with parameters n and β. Thus

$$E(n\hat{\beta}) = n\beta \text{ and } E(\hat{\beta}) = \beta \quad \blacksquare$$

Since $\hat{\beta}$ is a function of complete, sufficient statistics, it follows by an application of the Lehman-Scheffe theorem that $\hat{\beta}$ is a unique minimum variance unbiased estimator of β.

The maximum likelihood estimators have desirable properties particularly for large samples. Among others, the maximum likelihood estimator of a parameter is asymptotically normal and efficient. The main disadvantage however, is often the complex equations that are to be solved in finding the likelihood estimators.

3. The Cumulative Distribution Function Technique

Let $F_n(t)$ be the sample distribution function of t for a sample of size n. That is:

$$F_n(t)=\sum \text{sgn}\,(t-t_i)/n \tag{7}$$

where $\text{sgn}\,(t) = 0, \text{ if } t<0$
$\qquad\qquad = 1, \text{ if } t \geq 0.$

Then we need to solve:

$$(1/\beta)\int_{\alpha}^{t_\delta=4.5}(\exp\{-(t-\alpha)\}/\beta)dt=\delta \tag{8}$$

$$\frac{1}{\beta}\int_{4.5}^{t_{\delta+\omega}}\exp(-(t-\alpha)/\beta)dt=\delta+\omega \tag{9}$$

Expressions (8) and (9) effectively break the data into two parts to estimate α and β. These can be solved using the Newton-Rhapson method or any convenient numerical method.

4. Properties

We will now explore the properties of our estimators. We begin by assuming that α is fixed while $t_{(\alpha)}$ is a random variable. The following results are standard in Mathematical Statistics.

Lemma 1: If $(T_n-\theta)\xrightarrow{L}N(0,\delta^2)$, then $\sqrt{n}\,(f(T_n)-f(\theta))\xrightarrow{L}N(0,(f'(\theta)^2)$ provided f'(θ) exists and is not zero.

Proof: (Lehmann, 1983, p337)

Lemma 2: Let K_n be a sequence of integers such that $K_{n/n}=\alpha+Rn$ (0<α<1) and $\sqrt{n}\,Rn\to 0$ and let $X_1, X_2,\dots,X_n$ be iid with distribution f for which F(ξ) = α. Assume that F' = f exists and is positive at ξ.

Then:

$$\sqrt{n}\left(X_{(K_n)}-\xi\right)\longrightarrow N(0,\alpha(1-\alpha)/f^{2}(\xi))$$

Lemma 2 gives the asymptotic distribution of the α^{th} order statistics from equation (4).

Let G ($t_{(\alpha)}$,α) = F^{-1}($t_{(\alpha)}$,α). We know from Lemma 2 that $\sqrt{n}\left(t_{(\alpha)}-t_\alpha\right)\longrightarrow N(0,\alpha(1-\alpha)/f^{2}(t_{(\alpha)}))$. Now, $\sqrt{n}$ (θ-θ) = $\sqrt{n}$ ($F^{-1}(t_{(\alpha)})$ - F^{-1} (ω)) and by Lemma 1, $\sqrt{n}$ ($F^{-1}(t_{(\alpha)})$ - F^{-1} ($t_{(\alpha)}$))→N (0, α (1-α)/ $f(t_{(\alpha)}).(G'(t_{(\alpha)}))^2$). It remains to find an explicit expression for G'.

Theorem 3: Let $t_1,\dots,t_n$ be iid F(t; θ). Let 0<α<1 and suppose that $t_{(\alpha)}$ is the α^{th} population quantile. Let f = f' exists with $f(t_{(\alpha)})>0$, Let θ = F^{-1} ($t_{(\alpha)}$),

Then

$$\sqrt{n}\ (F^{-1}(t_{(\alpha)})\to N\ (0,\ \alpha\ (1-\alpha)/f^2(t_{(\alpha)}).(G'(t_{(\alpha)}))^2),\ \text{where } G'(t_{(\alpha)})=F^{-1}(t_{(\alpha)})$$

Proof: Apply Lemma 1 and Lemma 2.

Application: Let α be given and let

$$f(t) = (1/\beta)e^{-t/\beta}, \quad \beta \geq 0, t \geq 0.$$

Let $t_1,\ldots,t_n$ be iid F(t) and $t_{(\alpha)}$ be the α^{th} order statistics. Then it is easy to see that

$$\beta = -t_{(\alpha)} / \ln(1 - \alpha).$$

The derivative of f^1 is $G' = -1 / \ln(-\alpha)$. From Theorem 3, we obtain:

$$\sqrt{n}\,(\beta - t_{(\alpha)} / \ln(1 - \alpha) \rightarrow N(0, \beta^2\alpha(1-\alpha) / [\ln(1-\alpha)^2 e^{-2t\alpha/\beta})$$

4. Multiparameter Case

In the multiparameter case, the observed data is divided into p blocks to estimate the p parameters $\theta_1, \theta_2, \ldots, \theta_p$. For p = 2, the estimators may be from:

$$F(t_{(\alpha)}, \theta_1, \theta_2) = \alpha, \text{ and} \tag{10}$$

$$F(t_{\alpha+\delta}, \theta_1, \theta_2) = \alpha + \delta \text{ where } 0 < \alpha < 1, 0 < \delta < 1 \tag{11}$$

and $t_{(\alpha)}$ and $t_{\alpha+\delta}$ are the corresponding α^{th} and $(\alpha + \delta)^{th}$ quantiles.

The following Lemma about the asymptotic distribution of $t_{(\alpha)}$ and $t_{\alpha+\delta}$ may be employed to establish the asymptotic distribution of θ_1 and θ_2.

Lemma 3: Under the assumption of Lemma 2, let m_n be a sequence of integers such that $m_n/n = x + R'_n$, $0 < \alpha < \lambda < 1$, $\sqrt{n}\,R_n \rightarrow 0$, $F(\omega) = \lambda$ and $f(\omega) > 0$.

Then:

$$(\sqrt{n}\,(X'_{kn} - \xi), \sqrt{n}\,(X_{mn} - \omega))$$

is a Bivariate normal, vector with zero mean vector and covariance matrix $\Sigma = (\alpha_{ij})$ where:

$$\alpha_{11} = \alpha_{(1-\alpha)}/f^2(\xi), \qquad \alpha_{22} = \lambda(1-\lambda)/f^2(\omega), \qquad \alpha_{12} = \alpha_{21} = \alpha(1-\lambda)/f^2(\xi)(\omega)$$

Proof: (Lehmann, p354)

The asymptotic distribution of θ_1 and θ_2 may now be established by noting that:

$$\theta_1 = F^{-1}(t_\alpha, \theta_2, \alpha) \tag{12}$$

$$\theta_2 = F^{-1}(t_{\alpha+\delta}, F^{-1}(t_\alpha, \theta_2, \alpha), \alpha + \delta \tag{13}$$

and applying Lemma 1 and Lemma 3.

5. Observations

The efficiency of the estimators decreases as the number of parameters increases. Thus, it may be worthwhile trying a mixed procedure by combining the maximum likelihood procedure and the present technique. For the displaced exponential distribution, the MLE of α, $\hat{\alpha}$ = min $\{t_i\}$, may be used then β can be calculated using the cdf technique. In particular, this means that:

$$(1/\beta)\int_{\alpha}^{t_{\alpha}} e^{-(t-\alpha)/\beta} dt = \delta \quad (14)$$

needs to be solved. An ad hod procedure for solving this integral equation is to put α thus yielding:

$$\beta = (t_{(\delta)} - \alpha)/\ln(1 - \delta) \quad (15)$$

in many practical applications, the simplifying assumption that $\hat{\alpha} = \alpha$ will suffice.

6. Simulation

A small Monte Carlo experiment was conducted using a displaced exponential distribution with δ = 0.95, α = 1, β = 3. Thirty simulation runs were done using MICROSTAT. Each simulation was of length n = 100. The estimates and the corresponding biases and MSE are shown in Table 1.

7. Conclusion

Analysis of the mixed MLE-cdf technique revealed that the procedure is indeed promising. The Monte Carlo experiment confirmed that the variance and biases were of the order $0(1/n^2)$. The cdf-technique for the one-parameter case will be most useful when the cdf can be found in closed form (hence its inverse is easily found). In the multiparameter case, the location parameters may first be estimated by the MLE until one parameter is left which could then be estimated by the cdf-technique. It is found that the resulting estimate of the distribution function fits the observed data more closely. The maximum likelihood estimator may be negative in some instances which make their use impractical in the analysis of real-life data (Ashton, 1975).

8. References

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Table 1
Simulating the MLE-CDF Technique for the
Displaced Exponential: α = 1, β = 3

Run	$\hat{\alpha}$	$\hat{\beta}$	t	|Bias $\hat{\alpha}$|	|Bias $\hat{\beta}$|	ΣBias^2
1	1.02	3.33	11.00	.02	.33	.1093
2	1.04	2.99	10.00	.04	.01	.0017
3	1.02	2.67	9.00	.01	.33	.1090
4	1.09	3.00	10.09	.09	.00	.0081
5	1.04	3.06	10.21	.04	.06	.0052
6	1.03	3.06	10.30	.03	.06	.0045
7	1.05	2.99	10.00	.05	.01	.0026
8	1.02	2.99	10.00	.02	.01	.0005
9	1.19	3.27	11.00	.19	.27	.1090
10	1.12	3.15	10.55	.12	.15	.0369
11	1.09	3.03	10.06	.09	.03	.0090
12	1.06	2.98	10.00	.06	.02	.0040
13	1.19	3.01	10.21	.19	.01	.0362
14	1.04	2.99	10.00	.04	.01	.0017
15	1.02	3.01	10.05	.02	.01	.0005
16	1.03	3.01	10.06	.03	.01	.0010
17	1.06	3.00	10.06	.06	.00	.0036
18	1.03	3.06	10.21	.03	.06	.0045
19	1.01	3.00	10.00	.01	.00	.0001
20	1.02	2.99	10.00	.02	.01	.0005
21	1.04	3.00	10.05	.04	.00	.0016
22	1.06	3.00	10.06	.06	.00	.0036
23	1.03	2.98	9.95	.03	.02	.0013
24	1.04	3.00	10.05	.04	.00	.0016
25	1.14	3.29	11.00	.14	.29	.1037
26	1.09	2.99	10.05	.09	.01	.0082
27	1.02	2.99	10.00	.02	.01	.0005
28	1.02	2.99	10.00	.02	.01	.0005
29	1.03	3.01	10.05	.03	.01	.0010
30	1.04	3.00	10.05	.04	.00	.0016
Mean	1.06	3.03	-	.06	.06	-
SD	.05		MSE	.0190		